

AD-A087 006

NAVAL RESEARCH LAB WASHINGTON DC

F/G 12/1

DERIVATION OF THE VARIABLE COEFFICIENT KORTEWEG-DE VRIES EQUATION-ETC(U)

JUL 80 M S CRAMER

UNCLASSIFIED

NRL-MR-4222

SBIE-AD-E000 489

NL

1 OF 1
AD-A
000 000

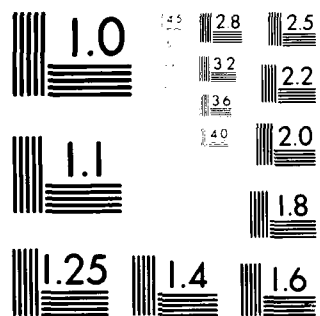
END

DATE

FILED

9-80

DTIC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

ADA 087006

ALL COPY

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

9 REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER NRL Memorandum Report 4222	2. GOVT ACCESSION NO. AD-A087006	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) DERIVATION OF THE VARIABLE COEFFICIENT KORTEWEG-DE VRIES EQUATION USING MULTITIMING TECHNIQUES		5. TYPE OF REPORT & PERIOD COVERED
6. AUTHOR(s) Mark S. Cramer		6. PERFORMING ORG. REPORT NUMBER
7. PERFORMING ORGANIZATION NAME AND ADDRESS Code 4340 Environmental Sciences Division Naval Research Laboratory, Washington, DC 20375		8. CONTRACT OR GRANT NUMBER(s) 16 RR03143
9. CONTROLLING OFFICE NAME AND ADDRESS Naval Research Laboratory Washington, DC 20375		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153 RR0314341 83-1140-0-0
11. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 1219		12. REPORT DATE 7 July 1980
		13. NUMBER OF PAGES 18
		14. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 19 SE-E		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) 19 NRL-MR-4222 19 AD-E000 419		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) water waves Korteweg-de Vries Equation multitiming shoaling waves		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The method of multiple scales or multitiming technique is used to derive the variable coefficient Korteweg de Vries (KdV) equation. Three small and independent parameters are introduced, ϵ which gives a measure of the amplitude of the wave, μ which gives a measure of the shallowness of the water and therefore dispersive effects and δ which gives a measure of the modulation rate due to variations in the undisturbed depth of the water. The range of validity of the variable coefficient KdV equation is given in terms of these parameters and the propagation distance of the wave.		

DD FORM 1473

1 JAN 73

EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

251751

43

CONTENTS

1. Introduction	1
2. Mathematical Formulation	2
3. Derivation of the Variable Coefficient KdV Equation	5
4. Discussion	12
REFERENCES	13
APPENDIX A - Equations Resulting from Perturbation Expansion	14

ACCESSION for		
NTIS	White Section	<input checked="" type="checkbox"/>
DDC	Buff Section	<input type="checkbox"/>
UNANNOUNCED		<input type="checkbox"/>
JUSTIFICATION		
BY		
DISTRIBUTION/AVAILABILITY CODES		
Dist.	AVAIL. and/or	SPECIAL
A		

DERIVATION OF THE VARIABLE COEFFICIENT KORTEWEG-de VRIES EQUATION USING MULTITIMING TECHNIQUES

1. Introduction

In the theory of shallow water waves the Korteweg-de Vries (KdV) equation

$$\eta_t + \frac{3}{2} \Sigma \eta \eta_x + \frac{1}{6} \eta_{xxx} = 0$$

plays a fundamental role. This equation was first derived by Korteweg and de Vries (1895) and describes the motion due to small amplitude waves in shallow water of uniform undisturbed depth. By shallow water we mean that the depth of the water is small compared to some characteristic length of the waves. The nondimensional parameter Σ is the Ursell number and is defined by

$$\Sigma \equiv \frac{a \lambda^2}{h_0^3},$$

where a is the maximum amplitude of the wave, λ is a characteristic length scale of the wave and h_0 is the undisturbed depth of the water. The importance of the KdV equation is that it can describe slightly dispersive linear wavetrains ($\Sigma \ll 1$), nonlinear long waves ($\Sigma \gg 1$) or cnoidal and solitary waves $\Sigma = O(1)$. Although the KdV equation cannot describe all the phenomena associated with shallow water waves, it nevertheless is an extremely useful tool in the study of such waves.

It is frequently of interest to determine the behavior of these waves as they move into regions of shallower and shallower water. The first study of this problem is due to Green (see Lamb (1932), Sect. 185). Green shows that when infinitely long linear waves travel over a gradually varying bottom the amplitude varies as $h^{-1/4}$ where $h \equiv h(x)$ is the undisturbed depth of the water. This amplification (or attenuation) law is often referred to as Green's law. Other authors (see, e.g., Grimshaw (1970) and Johnson (1973)) have studied the behavior of solitary waves as they propagate into water of decreasing depth. There it was shown that the wave amplifies like h^{-1} . Svendsen and Hansen (1978) have studied the behavior of cnoidal waves as they approach a beach and determined the Green's law appropriate for these waves. In each of these cases the wave was regarded as a slowly varying solitary, cnoidal or linear wave. In all of these studies but one, viz. Johnson (1973),* no attempt was made to derive an equivalent KdV equation for problems involving gradual depth changes. Johnson showed that the correct equation governing the surface elevation η is

$$\eta_x + \frac{3}{2} \epsilon h^{-3/2} \eta \eta_x + \mu^2 \frac{h^{1/2}}{6} \eta_{xxx} + \frac{\delta}{4} \frac{dh}{d\delta x} \frac{1}{h} \eta = 0$$

where x is the nondimensional distance measured parallel to direction of propagation, τ is a wave coordinate

$$\tau \equiv \frac{1}{\delta} \int_0^{\delta x} \frac{dp}{\sqrt{h(p)}} - t$$

Manuscript submitted March 10, 1980.

*The above equation for η has also been given without proof by Ostrovskiy and Pelinovskiy (1970).

where t is the nondimensional time and $h \equiv h(\delta x)$ is the undisturbed water depth scaled with some typical depth h_0 . Here ϵ , μ^2 , δ are small parameters defined later which measure the wave amplitude, the shallowness of the water and the gradualness of the bottom variation. It is clear that this equation is just the flat bottom KdV equation with slowly varying coefficients. Because of the importance of this equation to the study of waves in water of varying depth it is the intent of this paper to give a second derivation of this equation using the standard techniques of singular perturbation theory. The derivation of Johnson does not state the limits on the validity of the above equation nor is it clear for what relative values of the small parameters ϵ , μ^2 , δ the above equation holds.* Furthermore, the derivation presented by Johnson appears to preclude the possibility of reflected waves at the second order; see, for example, his expansion of η . Both higher order reflected and transmitted waves are known to exist. The reflected waves have been discussed by Peregrine (1967) and Madsen and Mei (1969). In the case of long linear waves Witting (private communication) has shown that reflected and transmitted waves are necessary for the satisfaction of certain conservation laws.

In the following report the variable coefficient KdV equation is derived by a straightforward application of the method of multiple scales or multitime. This technique has been discussed by many authors; excellent references are Cole (1968), Van Dyke (1975), Nayfeh (1974) and Leibovich and Seebass (1974). The analysis is conceptually straight-forward and because of the widespread use of the multitime technique it is felt that the derivation presented here is more accessible to the reader. Furthermore, the limits on the theory come immediately out of the analysis. In addition, the assumptions made allow for the possibility of reflected waves in the higher order terms.

2. Mathematical Formulation

We consider two dimensional motions of an inviscid fluid. The coordinate system is that sketched in Figure 1. The distance along the undisturbed free surface is \bar{x} and the distance measured normal to it is \bar{y} . The acceleration of gravity is in the negative \bar{y} direction and the bottom is given by $\bar{y} \equiv -h_0 h \left(\frac{\bar{x}}{L} \right)$, where $h(0) \equiv 1$ and the length scale L determines how fast the undisturbed fluid depth is changing. Because the fluid is regarded as inviscid we may assume a velocity potential $\bar{\phi} = \bar{\phi}(\bar{x}, \bar{y}, \bar{t})$ exists. In terms of this velocity potential the equation of motion and boundary conditions may be written:

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{y}\bar{y}} = 0 \text{ for } -h_0 h \left(\frac{\bar{x}}{L} \right) \leq \bar{y} \leq \bar{\eta}(\bar{x}, \bar{t})$$

$$\bar{\phi}_{\bar{y}} = -\frac{h_0}{L} \bar{\phi}_{\bar{x}} h' \left(\frac{\bar{x}}{L} \right) \text{ on } \bar{y} = -h_0 h \left(\frac{\bar{x}}{L} \right)$$

$$\bar{\phi}_{\bar{y}} = \bar{\eta}_{\bar{t}} + \bar{\phi}_{\bar{x}} \bar{\eta}_{\bar{x}} \text{ on } \bar{y} = \bar{\eta}$$

$$\bar{\eta} = -\frac{1}{g} \left\{ \bar{\phi}_{\bar{t}} + \frac{1}{2} (\bar{\phi}_{\bar{x}}^2 + \bar{\phi}_{\bar{y}}^2) \right\} \text{ on } \bar{y} = \bar{\eta}.$$

Here the free surface is given by $\bar{y} = \bar{\eta}(\bar{x}, \bar{t})$, where \bar{t} , of course, is the time. The first equation represents mass conservation at each point in the fluid, the second equation expresses the fact

*Such questions, while of general interest, are particularly important in the study of the singularity as $h \rightarrow 0$.

that the bottom, $\bar{y} = -h_0 h \left(\frac{\bar{x}}{L} \right)$, is impenetrable; here $h'(\xi) \equiv \frac{dh}{d\xi}$. The last two equations are the boundary conditions at the free surface $\bar{y} = \bar{\eta}$. Here g is the acceleration of gravity. The third is the kinematic condition at the free surface and the fourth is the Bernoulli equation evaluated at the free surface. Implicit in the derivation of the latter is the assumption that the pressure is constant at the surface; hence we are ignoring any surface tension and the effect of any fluid above the water.

To complete our description of the fluid motion we need an initial or boundary condition which, of course, produces the wave motion of the free surface. Fortunately, we do not need an explicit condition in order to derive the variable coefficient KdV equation. It will be sufficient to assume that this condition introduces an amplitude a and characteristic length or wavelength λ into the problem; for example, a typical initial condition might be $\bar{\eta} = a F \left(\frac{\bar{x}}{\lambda} \right)$ at $\bar{t} \equiv 0$. The initial or boundary condition will only enter the problem through a and λ ; these will be used for scaling purposes later.

We now nondimensionalize the equations of motion as follows

$$\begin{aligned}\bar{x} &= \lambda x \\ \bar{y} &= h_0 y \\ \bar{t} &= \frac{\lambda}{\sqrt{gh_0}} t \\ \bar{\eta} &= \epsilon h_0 \eta \\ \bar{\phi} &= \epsilon \lambda \sqrt{gh_0} \phi\end{aligned}$$

where x, y, t, η, ϕ are the new nondimensional variables and $\epsilon \equiv \frac{a}{h_0}$. The above equations become

$$\begin{aligned}\mu^2 \phi_{xx} + \phi_{yy} &= 0 \quad -h(\delta x) \leq y \leq \epsilon \eta \\ \phi_y &= -\mu^2 \delta \phi_x h'(\delta x) \quad \text{on } y = -h(x\delta) \\ \left. \begin{aligned} \phi_y &= \mu^2 [\eta_t + \epsilon \phi_x \eta_x] \\ -\eta &= \phi_t + \frac{1}{2} \epsilon \phi_x^2 + \frac{1}{2} \frac{\epsilon}{\mu^2} \phi_y^2 \end{aligned} \right\} \quad \text{on } y = \epsilon \eta,\end{aligned} \tag{1}$$

where $\mu \equiv \frac{h_0}{\lambda}$, $\delta \equiv \frac{\lambda}{L}$.

We shall now study waves whose amplitudes are small, i.e., $\epsilon \ll 1$. We shall also assume that the water is shallow, that is, that the lengths of the waves are large compared to the depth, i.e., $\mu \ll 1$. In addition, we will regard the bottom as slowly varying, that is, the lengths of the waves are taken to be short compared to L , i.e., $\delta \ll 1$. In terms of the physical parameters this requires $a \ll h_0 \ll \lambda \ll L$. At this stage we need not assume any particular ordering of ϵ, μ , and δ ; in this we differ from Johnson (1973) who at least at the outset takes $\epsilon = O(\mu^2) = O(\delta)$.

*It should be mentioned that we are also assuming that the initial or boundary condition is such that to lowest order the wave(s) move to the right, i.e., in the positive x direction

MARK S. CRAMER

Because we are taking ϵ , μ^2 and δ small, a natural approach to this problem is to seek perturbation solutions

$$\begin{aligned}\eta = \eta(x, t; \epsilon, \delta, \mu) = & \eta_0 + \mu^2 \eta_1 + \delta \eta_2 + \epsilon \eta_3 + \mu^4 \eta_4 + \mu^2 \delta \eta_5 \\ & + \delta^2 \eta_6 + \mu^2 \epsilon \eta_7 + \delta \epsilon \eta_8 + \epsilon^2 \eta_9 \\ & + \dots\end{aligned}$$

and

$$\begin{aligned}\phi = \phi(x, y, t; \epsilon, \delta, \mu) = & \phi_0 + \mu^2 \phi_1 + \delta \phi_2 + \epsilon \phi_3 + \mu^4 \phi_4 + \mu^2 \delta \phi_5 \\ & + \delta^2 \phi_6 + \mu^2 \epsilon \phi_7 + \delta \epsilon \phi_8 + \epsilon^2 \phi_9 \\ & + \dots,\end{aligned}$$

where the dots indicate terms of third order in ϵ , δ and μ^2 and $\eta_i = \eta_i(x, t)$ only and $\phi_i = \phi_i(x, y, t)$ only, $i = 0, 1, 2, 3, \dots$. Although we are regarding ϵ , δ and μ^2 as arbitrarily small these expansions are seen to break down at large values of x and t . In particular, it can be shown that η_1 contains terms which are proportional to x ; thus when $x = O\left(\frac{1}{\mu^2}\right)$ the $\mu^2 \eta$ term is just as large as the η_0 term and we no longer have a valid asymptotic expansion. Nonuniformities such as these are well known and techniques have been devised which can render the above expansion valid for large values of x and t . One of these techniques is called the method of multiple scales or the multitiming technique; see Cole (1968), Van Dyke (1974), Nayfeh (1974) or Leibovich and Seebass (1974). In fact, both Cole and Leibovich and Seebass have used this technique to derive the flat bottom or constant coefficient version of the KdV equation, suggesting that this is a natural way to derive the variable coefficient equation. The basic approach of the multitiming technique is to regard $\eta = \eta(x, t; \epsilon, \delta, \mu^2)$ as a function of new independent variables

$$\begin{aligned}\tau &\equiv \frac{1}{\delta} g(x_2) - t, \\ x_0 &\equiv x, \\ x_1 &\equiv \mu^2 x, \\ x_2 &\equiv \delta x,\end{aligned}\tag{2}$$

and

$$x_3 \equiv \epsilon x,$$

i.e., $\eta = \eta(\tau, x_0, x_1, x_2, x_3; \mu^2, \delta, \epsilon)$. Here τ is the wave coordinate previously introduced. This allows for the change in the wave speed due to the changing depth; in theories where there is no change in the propagation speed τ would simply be $\tau = x - t$ or $g(x_2) \equiv x_2 \equiv \delta x$. In the multitiming technique τ , x_0 , x_1 , x_2 , and x_3 are all regarded as independent variables.

When $x_0 = x = O\left(\frac{1}{\delta}\right)$, $\tau = O(1)$ and $x_2 = O(1)$ and when $x_0 = x = O\left(\frac{1}{\mu^2}\right)$, $x_1 = O(1)$ and when $x_0 = x = O\left(\frac{1}{\epsilon}\right)$, $x_3 = O(1)$. With this assumption we then go through the perturbation procedure just as before. However, we require that our expansion be valid for $x = O\left(\frac{1}{\mu^2}, \frac{1}{\epsilon}, \text{ and } \frac{1}{\delta}\right)$; it turns out that this requirement gives the dependence of η_0 on x_1 , x_2 , x_3 and ultimately the variable coefficient KdV equation.

One motivation for this technique is found by examining the failure of the straightforward expansion. That expansion is seen to have the following form

$$\eta = \eta_0 + \mu^2 [x \cdot \text{const.} + f(x)] + \dots$$

where both η_0 and f are bounded in x . This would seem to suggest that the exact solution for η can be written

$$\eta = \eta(x, t; \epsilon, \mu^2, \delta) = \eta(x, t, \mu^2 x, \mu^2 t; \epsilon, \mu^2, \delta).$$

Similar remarks apply for ϵ and δ . Another more intuitive explanation is that we see two simultaneous changes in x . The first is due to the fact that the wave carries changes in the water height and in the fluid velocity. By definition of λ these changes occur over a distance $x = O(1)$. In a coordinate system traveling with the wave these are seen as variations with respect to τ . The second is that the basic wave shape is being slowly modified due to non-linearity, dispersion and the bottom variation. Thus, it would seem reasonable to express η in the above manner. Of course, multitime techniques are well established in mechanics and applied mathematics and these comments are not meant to be a complete justification of the technique, rather they are meant to remind the reader of some of the intuitive ideas connected with this method. For a more complete discussion of multitime techniques the reader is referred to the above mentioned references.

3. Derivation of the Variable Coefficient KdV Equation

In our derivation we first need to expand equations (1) in powers of ϵ , δ and μ^2 . In the following we may regard ϵ , δ and μ^2 as independent and of any relative size. However, as in any multitime scheme involving more than one small parameter there will be difficulties with this viewpoint, see, for example, Chapter IV of Leibovich and Seebass (1974). Because these are most easily seen by an analysis of higher order terms not discussed below, we shall delay the discussion of this until Section 4. It will be noted at this stage that these difficulties are to be expected in any multitime approach and of course are implicit in Johnson's (1973) work.

We shall first apply the basic assumption of multitime, namely that the independent variables are given by (2). We do this in a stepwise fashion; first we change variable from x and t to x and τ , where $\tau \equiv \frac{1}{\delta} g(\delta x) - t$ and $x = x$. The x and t derivatives can then be transformed using the chain rule

$$\begin{aligned} \left. \frac{\partial}{\partial t} \right|_x &= - \left. \frac{\partial}{\partial \tau} \right|_x \\ \left. \frac{\partial}{\partial x} \right|_t &= \left. \frac{\partial}{\partial x} \right|_\tau + \dot{g} \left. \frac{\partial}{\partial \tau} \right|_x \end{aligned}$$

and

$$\frac{\partial^2}{\partial x^2} = \dot{g}^2 \frac{\partial^2}{\partial \tau^2} + 2\dot{g} \frac{\partial^2}{\partial x \partial \tau} + \frac{\partial^2}{\partial x^2} + \delta \ddot{g} \frac{\partial}{\partial \tau},$$

where $\dot{g}(\xi) \equiv \frac{dg}{d\xi}$. We further transform x and τ to the independent variables x_0, x_1, x_2, x_3 and τ ; again by the chain rule

$$\begin{aligned} \left. \frac{\partial}{\partial t} \right|_x &= - \left. \frac{\partial}{\partial \tau} \right|_x \\ \left. \frac{\partial}{\partial x} \right|_t &= \dot{g} \left. \frac{\partial}{\partial \tau} \right|_x + \left. \frac{\partial}{\partial x_0} \right|_t + \mu^2 \left. \frac{\partial}{\partial x_1} \right|_t + \delta \left. \frac{\partial}{\partial x_2} \right|_t + \epsilon \left. \frac{\partial}{\partial x_3} \right|_t \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \Bigg| &= \dot{g}^2 \frac{\partial^2}{\partial \tau^2} + 2\dot{g} \frac{\partial^2}{\partial x_0 \partial \tau} + \frac{\partial^2}{\partial x_0^2} \\ &+ \delta \left[\ddot{g} \frac{\partial}{\partial \tau} + 2\dot{g} \frac{\partial^2}{\partial x_2 \partial \tau} + 2 \frac{\partial^2}{\partial x_0 \partial x_2} \right] \\ &+ 2\mu^2 \left[\dot{g} \frac{\partial^2}{\partial x_1 \partial \tau} + \frac{\partial^2}{\partial x_1 \partial x_0} \right] \\ &+ 2\epsilon \left[\dot{g} \frac{\partial^2}{\partial x_3 \partial \tau} + \frac{\partial^2}{\partial x_3 \partial x_0} \right] \\ &+ O(\mu^4, \mu^2\delta, \mu^2\epsilon, \delta^2, \delta\epsilon, \epsilon^2), \end{aligned}$$

here, of course, $g \equiv g(x_2)$ and in (1) we will take $h = h(x\delta) = h(x_2)$. Equations (1) therefore become

$$\begin{aligned} \phi_{yy} &= -\mu^2 [\dot{g}^2 \phi_{\tau\tau} + 2\dot{g} \phi_{x_0\tau} + \phi_{x_0x_0}] - \mu^2 \delta [\ddot{g} \phi_\tau + 2\dot{g} \phi_{x_2\tau} + 2\phi_{x_0x_2}] \\ &- 2\mu^4 [\dot{g} \phi_{x_1\tau} + \phi_{x_0x_1}] - 2\mu^2 \epsilon [\dot{g} \phi_{x_3\tau} + \phi_{x_3x_0}] \\ &+ O(\mu^6, \mu^4\delta, \mu^4\epsilon, \mu^2\delta^2, \mu^2\delta\epsilon, \mu^2\epsilon^2), \end{aligned} \quad (2a)$$

and

$$\phi_y = -\mu^2 \delta [\dot{g} \phi_\tau + \phi_{x_0}] h'(x_2) + o(\mu^2 \delta) \text{ on } y = -h(x_2) \quad (2b)$$

and

$$\phi_y = -\mu^2 \eta_\tau + \epsilon \mu^2 (\dot{g} \eta_\tau + \eta_{x_0}) (\dot{g} \phi_\tau + \phi_{x_0}) + o(\epsilon \mu^2) \text{ on } y = \epsilon \eta, \quad (2c)$$

and

$$\begin{aligned} -\eta &= -\phi_\tau + \frac{1}{2} \epsilon (\dot{g} \phi_\tau + \phi_{x_0})^2 \\ &+ \epsilon \mu^2 \phi_{x_1} (\dot{g} \phi_\tau + \phi_{x_0}) + \epsilon \delta \phi_{x_2} (\dot{g} \phi_\tau + \phi_{x_0}) \\ &+ \epsilon^2 \phi_{x_3} (\dot{g} \phi_\tau + \phi_{x_0}) + O(\epsilon \mu^4, \delta^2 \epsilon, \epsilon^3) \\ &+ \frac{1}{2} \frac{\epsilon}{\mu^2} \phi_y^2, \text{ on } y = \epsilon \eta. \end{aligned} \quad (2d)$$

At the next stage of the analysis we would need to expand ϕ in the two boundary conditions at $y = \epsilon \eta$, eqns (2)c,d on $y = \epsilon \eta$ we may write

$$\phi = \phi(x, t, y) = \phi(x, t, \epsilon \eta).$$

Hence for small ϵ this may be expanded in a Taylor series about $y = 0$:

$$\phi = \phi(x, t, 0) + \epsilon \eta \phi_y(x, t, 0) + \dots$$

In like manner the derivatives $\phi_\tau, \phi_{x_0}, \phi_{x_1}, \phi_{x_2}, \phi_{x_3}, \phi_y$ appearing in (2)c,d may also be expanded

$$\begin{aligned}\phi_\tau(\epsilon\eta) &= \phi_\tau(0) + \epsilon\eta \phi_{y\tau}(0) + \dots, \\ \phi_{x_0}(\epsilon\eta) &= \phi_{x_0}(0) + \epsilon\eta \phi_{yx_0}(0) + \dots, \\ \phi_{x_1}(\epsilon\eta) &= \phi_{x_1}(0) + \epsilon\eta \phi_{yx_1}(0) + \dots, \\ \phi_{x_2}(\epsilon\eta) &= \phi_{x_2}(0) + \epsilon\eta \phi_{yx_2}(0) + \dots, \\ \phi_{x_3}(\epsilon\eta) &= \phi_{x_3}(0) + \epsilon\eta \phi_{yx_3}(0) + \dots,\end{aligned}\tag{3}$$

and

$$\phi_y(\epsilon\eta) = \phi_y(0) + \epsilon\eta \phi_{yy}(0) + \dots,$$

Ordinarily we would simply substitute these expansions in the boundary conditions (2)c, d. However, to save space we shall anticipate a result which comes out of the analysis presented later. This is that $\phi_y(\tau, x_0, x_1, \dots, y) = O(\mu^2)$ at most. We now use this result and substitute expansions (3) in equations (2)c and (2)d; these boundary conditions then become

$$\phi_y = -\epsilon\eta \phi_{yy} - \mu^2 \eta_\tau + \epsilon\mu^2 (\dot{g}\eta_\tau + \eta_{x_0}) (\dot{g}\phi_\tau + \phi_{x_0}) + o(\epsilon\mu^2), \text{ on } y = 0,\tag{4}$$

and

$$\begin{aligned}-\eta &= -\phi_\tau - \epsilon\eta \phi_{y\tau} + \frac{1}{2} \epsilon (\dot{g}\phi_\tau + \phi_{x_0})^2 + \epsilon\mu^2 \phi_{x_1} (\dot{g}\phi_\tau + \phi_{x_0}) \\ &\quad + \epsilon\delta \phi_{x_2} (\dot{g}\phi_\tau + \phi_{x_0}) + \epsilon^2 \phi_{x_3} (\dot{g}\phi_\tau + \phi_{x_0}) \\ &\quad + O(\epsilon^2 \mu^2, \epsilon\mu^4, \delta^2 \epsilon, \epsilon^3) + \frac{1}{2} \frac{\epsilon}{\mu^2} \phi_y^2, \\ &\text{on } y = 0.\end{aligned}\tag{5}$$

Because we are anticipating the result $\phi_y = O(\mu^2)$ we see that the main effect due to the expansions (3) is the addition of a $-\epsilon\eta \phi_{yy}$ term to (2c) and $-\epsilon\eta \phi_{y\tau}$ term to (2d) and the transfer of the boundary condition from the moving surface $y = \epsilon\eta(x, t)$ to the stationary surface $y = 0$, at least to the order considered here. The surface elevation η and the potential ϕ are now given by the equations (2a), (2b), (4) and (5).

Finally we need to expand

$$\eta = \eta(\tau, x_0, x_1, x_2, x_3; \epsilon, \mu^2, \delta),$$

and

$$\phi = \phi(\tau, x_0, x_1, x_2, x_3; \epsilon, \mu^2, \delta)$$

in expansions of powers of ϵ, μ^2 and δ , i.e.,

$$\begin{aligned}\eta &= \eta_0 + \mu^2 \eta_1 + \delta \eta_2 + \epsilon \eta_3 + \mu^4 \eta_4 + \mu^2 \delta \eta_5 + \delta^2 \eta_6 + \mu^2 \epsilon \eta_7 + \delta \epsilon \eta_8 + \epsilon^2 \eta_9 + \dots \\ \phi &= \phi_0 + \mu^2 \phi_1 + \delta \phi_2 + \epsilon \phi_3 + \mu^4 \phi_4 + \mu^2 \delta \phi_5 + \delta^2 \phi_6 + \mu^2 \epsilon \phi_7 + \delta \epsilon \phi_8 + \epsilon^2 \phi_9 + \dots\end{aligned}$$

Here the dots refer to terms of third order in μ^2, δ and ϵ . When these expansions are substituted in equations (2a), (2b), (4) and (5), and terms of like order in $\mu^2, \delta, \epsilon, \mu^4, \mu^2 \delta, \dots$

MARK S. CRAMER

are equated, we obtain a system of equations which need to be solved; these are listed in the Appendix. We now proceed to solve these keeping in mind that we want to eliminate any secular, i.e., proportional to x_0 , terms which may occur in η_i or ϕ_i .

From equations (A1) we see that $\phi_0 = B_0(\tau, x_0, x_1, x_2, x_3)$ only. We will also require that the wave(s) move in the positive x direction only and will therefore assume that B_0 is independent of x_0 , i.e.,

$$\phi_0 = B_0(\tau, x_1, x_2, x_3),$$

and

$$\eta_0 = B_{0\tau}.$$

We now consider equations (A2); these now read

$$\phi_{1yy} = -\dot{g}^2 B_{0\tau\tau},$$

$$\phi_{1y} = 0 \text{ on } y = -h,$$

and

$$\phi_{1y} = -B_{0\tau\tau} \text{ on } y = 0.$$

The first equation implies

$$\phi_{1y} = -\dot{g}^2 B_{0\tau\tau} y + A_1(\tau, x_0, x_1, x_2, x_3)$$

Substitution in the boundary condition at $y = -h$ yields A_1 :

$$A_1 = -\dot{g}^2 h B_{0\tau\tau},$$

hence

$$\phi_{1y} = -\dot{g}^2 B_{0\tau\tau} (y + h) \text{ in general.}$$

We also need to satisfy the boundary condition at $y = 0$; this then requires that

$$\dot{g}^2 \equiv \frac{1}{h(x_2)},$$

or upon integration

$$g(x_2) = \int_0^{x_2} h^{-1/2}(p) dp,$$

where we have chosen the positive sign for g . This of course agrees with the result presented earlier for τ . Thus, our results for ϕ_1 and η_1 are

$$\phi_1 = -\frac{1}{h} \left(\frac{y^2}{2} + hy \right) B_{0\tau\tau} + B_1(\tau, x_0, x_1, x_2, x_3), \quad (6)$$

and

$$\eta_1 = B_{1\tau}.$$

We now consider equations (A3). Integration and application of the boundary conditions at $y = -h$ and $y = 0$ show that $\phi_{2y} \equiv 0$ and therefore

$$\phi_2 = B_2(\tau, x_0, x_1, x_2, x_3),$$

and from the last equation in (A3)

$$\eta_2 = B_{2\tau}.$$

We now consider equations (A4). As in equations (A3) we can easily show that

$$\phi_3 = B_3(\tau, x_0, x_1, x_2, x_3)$$

and

$$\eta_3 = B_{3\tau} - \frac{1}{2h} \eta_0^2.$$

When we use the fact that $\phi_{ox_0} \equiv 0$ and our results for ϕ_1 , equations (A5) become

$$\phi_{4yy} = -\left[\frac{1}{h} B_{1\tau\tau} + \frac{1}{\sqrt{h}} B_{1x_0\tau} + B_{1x_0x_0}\right] + \frac{1}{h^2} \left(\frac{y^2}{2} + hy\right) \eta_{0\tau\tau\tau} - \frac{2}{\sqrt{h}} \eta_{0x_1}.$$

$$\phi_{4y} = 0 \text{ on } y = -h$$

$$\phi_{4y} = -B_{1\tau\tau} \text{ on } y = 0.$$

If we integrate the first equation we find

$$\begin{aligned} \phi_{4y} = & -\left[\frac{1}{h} B_{1\tau\tau} + \frac{2}{\sqrt{h}} B_{1x_0\tau} + B_{1x_0\tau} + B_{1x_0x_0}\right] y + \frac{1}{h^2} \left(\frac{y^3}{6} + \frac{hy^2}{2}\right) \eta_{0\tau\tau\tau} \\ & - \frac{2}{\sqrt{h}} \eta_{0x_1} y + A_4(\tau, x_0, x_1, x_2, x_3). \end{aligned}$$

The boundary condition at $y = -h$ determines A_4 as

$$A_4 = -2\sqrt{h} \eta_{0x_1} - \frac{h}{3} \eta_{0\tau\tau\tau} - h \left[\frac{1}{h} B_{1\tau\tau} + \frac{2}{\sqrt{h}} B_{1x_0\tau} + B_{1x_0x_0} \right].$$

The boundary condition at $y = 0$ gives us the equation for B_1 :

$$A_4 = -B_{1\tau\tau}$$

or

$$\frac{2}{\sqrt{h}} B_{1x_0\tau} + B_{1x_0x_0} = -\left[2\sqrt{h} \eta_{0x_1} + \frac{h}{3} \eta_{0\tau\tau\tau}\right].$$

We note that a particular solution for B_1 is

$$B_1 = -x_0 \frac{\sqrt{h}}{2} \int K_1 d\tau$$

or

$$\eta_1 = B_{1\tau} = -x_0 \frac{\sqrt{h}}{2} K_1,$$

where $K_1 = K_1(\tau, x_1, x_2, x_3) \equiv 2\sqrt{h} \eta_{0x_1} + \frac{h}{3} \eta_{0\tau\tau\tau}$. Thus, η_1 grows linearly with x_0 and when $x_0 = x = 0\left(\frac{1}{\mu^2}\right)$ our expansion scheme breaks down, i.e., $\mu^2 \eta_1 = 0(\eta_0)$, unless $K_1 \equiv 0$, i.e.,

MARK S. CRAMER

$$\eta_{ox_1} + \frac{\sqrt{h}}{6} \eta_{o\tau\tau\tau} = 0. \quad (7)$$

Thus, in order to avoid the breakdown of our expansion when $x = O\left(\frac{1}{\mu^2}\right)$ we shall assume that the x_1 behavior of η_o is given by (7). We note that there is no assumption here on B_1 , except, of course, that $B_{1\tau} = \eta_1$ be such that our expansion be valid for large values of x . A more careful analysis of the homogeneous solution for B_1 shows that both right and left moving waves are possible. Had we assumed $B_{1x_0} \equiv 0$ or $\eta_{1x_0} \equiv 0$ as does Johnson (1973), this would not be the case.

We now consider equations (A6) which now read

$$\phi_{5yy} = -\left[\frac{1}{h} B_{2\tau\tau} + \frac{2}{\sqrt{h}} B_{2x_0\tau} + B_{2x_0x_0}\right] + \frac{1}{2} h^{-\frac{3}{2}} h' \eta_o - \frac{2}{\sqrt{h}} \eta_{ox_2}$$

$$\phi_{5y} = -\frac{h'}{\sqrt{h}} \eta_o \text{ on } y = -h$$

$$\phi_{5y} = -B_{2\tau\tau} \text{ on } y = 0.$$

Integration of the first equation yields

$$\begin{aligned} \phi_{5y} = & -\left[\frac{1}{h} B_{2\tau\tau} + \frac{2}{\sqrt{h}} B_{2x_0\tau} + B_{2x_0x_0}\right] y + \frac{1}{2} h^{-\frac{3}{2}} h' \eta_o y \\ & - \frac{2}{\sqrt{h}} \eta_{ox_2} y + A_5(\tau, x_0, x_1, x_2, x_3) \end{aligned}$$

The first boundary condition yields A_5

$$A_5 = -\frac{1}{2} \frac{h'}{\sqrt{h}} \eta_o - 2\sqrt{h} \eta_{ox_2} - \left[\frac{1}{h} B_{2\tau\tau} + \frac{2}{\sqrt{h}} B_{2x_0\tau} + B_{2x_0x_0}\right] h.$$

The second boundary condition yields our equation for B_2 :

$$\begin{aligned} A_5 = & -B_{2\tau\tau} \\ \frac{2}{\sqrt{h}} B_{2x_0\tau} + B_{2x_0x_0} = & -\frac{1}{h} \left[2\sqrt{h} \eta_{ox_2} + \frac{1}{2} \frac{h'}{\sqrt{h}} \eta_o \right]. \end{aligned}$$

A particular solution for B_2 is

$$B_2 = -x_0 \frac{\sqrt{h}}{2} \int K_2 d\tau$$

or

$$\eta_2 = B_{2\tau} = -x_0 \frac{\sqrt{h}}{2} K_2,$$

where $K_2 \equiv \frac{1}{h} \left[2\sqrt{h} \eta_{ox_2} + \frac{1}{2} \frac{h'}{\sqrt{h}} \eta_o \right]$. Thus, when $x_0 = x = O\left(\frac{1}{\delta}\right)$, $\delta\eta_2 = O(\eta_o)$ unless $K_2 \equiv 0$. Thus, in order to avoid a breakdown in our expansion when $x = O\left(\frac{1}{\delta}\right)$ we shall require that $K_2 \equiv 0$, i.e.,

$$\eta_{ox_2} + \frac{1}{4} \frac{h'(x_2)}{h(x_2)} \eta_o = 0. \quad (8)$$

If we consider equations (A7) we can easily show that $\phi_6 = B_6(\tau, x_0, x_1, x_2, x_3)$ and $\eta_6 = B_{6\tau}$.

When our results for ϕ_0 , ϕ_1 and ϕ_3 are substituted in (A8) we have

$$\phi_{7yy} = -\left[\frac{1}{h} B_{3\tau\tau} + \frac{2}{\sqrt{h}} B_{3x_0\tau} + B_{3x_0x_0}\right] - \frac{2}{\sqrt{h}} \eta_{\alpha x_3}$$

$$\phi_{7y} = 0 \text{ on } y = -h$$

$$\phi_{7y} = \frac{3}{h} \eta_0 \eta_{0\tau} - B_{3\tau\tau} \text{ on } y = 0.$$

Integration of the first yields

$$\phi_{7y} = -\left[\frac{1}{h} B_{3\tau\tau} + \frac{2}{\sqrt{h}} B_{3x_0\tau} + B_{3x_0x_0}\right] y - \frac{2}{\sqrt{h}} \eta_{\alpha x_3} y + A_7(\tau, x_0, x_1, x_2, x_3).$$

Substitution into the first boundary conditions shows that A_4 is given by

$$A_7 = -\left[\frac{1}{h} B_{3\tau\tau} + \frac{2}{\sqrt{h}} B_{3x_0\tau} + B_{3x_0x_0}\right] h - 2\sqrt{h} \eta_{\alpha x_3}$$

The second boundary condition yields our equation for B_3 :

$$A_7 = -B_{3\tau\tau} + \frac{3}{h} \eta_0 \eta_{0\tau}$$

or

$$\frac{2}{\sqrt{h}} B_{3x_0\tau} + B_{3x_0x_0} = -\frac{1}{h} \left[2\sqrt{h} \eta_{\alpha x_3} + \frac{3}{h} \eta_0 \eta_{0\tau} \right].$$

A particular solution for B_3 is

$$B_3 = -x_0 \frac{\sqrt{h}}{2} \int K_3 d\tau$$

or

$$B_{3\tau} = -x_0 \frac{\sqrt{h}}{2} K_3,$$

where $K_3 \equiv \frac{1}{h} \left[2\sqrt{h} \eta_{\alpha x_3} + \frac{3}{h} \eta_0 \eta_{0\tau} \right]$. Thus, $\eta_3 = B_{3\tau} - \frac{1}{2} \frac{1}{h} \eta_0^2$ grows linearly with x_0 and when $x_0 = x = O\left(\frac{1}{\epsilon}\right)$ our expansion breaks down, i.e., $\epsilon \eta_3 = O(\eta_0)$, unless $K_3 \equiv 0$. Thus we shall require that $K_3 \equiv 0$, that is,

$$\eta_{\alpha x_3} + \frac{3}{2} h^{-\frac{3}{2}} \eta_0 \eta_{0\tau} \equiv 0. \quad (9)$$

If we continue on to equations (A9) and (A10) we see that

$$\phi_8 = B_8(\tau, x_0, x_1, x_2, x_3) \text{ only,}$$

and

$$\phi_9 = B_9(\tau, x_0, x_1, x_2, x_3) \text{ only.}$$

The functions η_8 and η_9 are of course given by the last equations in (A9) and (A10).

This completes our analysis of the perturbation equations. In order that our expansion be valid up to $x = O\left(\frac{1}{\mu^2}\right)$ or $O\left(\frac{1}{\epsilon}\right)$ or $O\left(\frac{1}{\delta}\right)$ we see that we need to modify the behavior of η_o at those large distances; this is the expected result of any multitiming scheme. The necessary modification to η_o are given by equations (7), (8), and (9) and are rewritten here for convenience

$$\begin{aligned}\frac{\partial \eta_o}{\partial x_1} &= -\frac{\sqrt{h}}{6} \frac{\partial^3 \eta_o}{\partial \tau^3} \\ \frac{\partial \eta_o}{\partial x_2} &= -\frac{1}{4} \frac{h'(x_2)}{h(x_2)} \eta_o\end{aligned}$$

and

$$\frac{\partial \eta_o}{\partial x_3} = -\frac{3}{2} h^{-\frac{3}{2}} \eta_o \frac{\partial \eta_o}{\partial \tau}.$$

We can show that η_o must satisfy the variable coefficient KdV equation by "reconstituting" the above equations, that is, by writing

$$\frac{\partial \eta_o}{\partial x} \bigg|_{\tau} = \frac{\partial \eta_o}{\partial x_0} \bigg|_{\tau, x_1, x_2, x_3} + \mu^2 \frac{\partial \eta_o}{\partial x_1} \bigg|_{\tau, x_0, x_2, x_3} + \delta \frac{\partial \eta_o}{\partial x_2} \bigg|_{\tau, x_0, x_1, x_3} + \epsilon \frac{\partial \eta_o}{\partial x_3} \bigg|_{\tau, x_0, x_1, x_2}.$$

This, of course, just follows from the chain rule and has already been used to transform from x, τ coordinates to the τ, x_0, x_1, x_2, x_3 coordinates. Thus,

$$\frac{\partial \eta_o}{\partial x} = 0 - \mu^2 \frac{\sqrt{h}}{6} \eta_{o\tau\tau\tau} - \frac{\delta}{4} \frac{h'}{h} \eta_o - \epsilon \frac{3}{2} h^{-\frac{3}{2}} \eta_o \eta_{o\tau},$$

or

$$\eta_{ox} + \mu^2 \frac{\sqrt{h}}{6} \eta_{o\tau\tau\tau} + \frac{\delta}{4} \frac{h'}{h} \eta_o + \frac{3}{2} \epsilon h^{-\frac{3}{2}} \eta_o \eta_{o\tau} = 0, \quad (10)$$

where we would now regard $h = h(\delta x)$. Once we allow for differences in notation we see that Equation (10) is exactly equation (14) of Johnson (1973) or equation (1) of Ostrovsky and Pelinovskiy (1970).

4. Discussion

We have derived the variable coefficient KdV equation (10) by standard multitiming procedures. This gives the correct behavior of η_o over distances and times of order $\frac{1}{\epsilon}$ or $\frac{1}{\mu^2}$ or $\frac{1}{\delta}$, whichever gives the largest distance. The behavior for larger distances, say $x = O\left(\frac{1}{\epsilon^2}\right)$ or $O\left(\frac{1}{\epsilon\mu^2}\right)$, would require an analysis of higher order terms and would show that we need to further correct η_o . Thus, some care should be exercised when one of the parameters is very much less than the other two. For example, if $\delta \ll \epsilon$ or μ^2 , say $\delta = \epsilon^2$, then (10) is not expected to give an accurate description of the flow for $x = O\left(\frac{1}{\delta}\right)$. We would then need to include the corrections necessary at $x = O\left(\frac{1}{\epsilon^2}\right)$. In this case (10) would reduce to

$$\eta_{ox} + \frac{\mu^2}{6} \eta_{o\tau\tau\tau} + \frac{3}{2} \epsilon \eta_o \eta_{o\tau} = 0,$$

and would only be valid for $x = O\left(\frac{1}{\epsilon}, \frac{1}{\mu^2}\right)$. One obvious way to avoid this is to require that ϵ , δ and μ^2 are all much larger than any quadratic term involving δ , ϵ and μ^2 . This would then ensure that the only corrections needed for $x = O\left(\frac{1}{\epsilon}, \frac{1}{\mu^2}, \frac{1}{\delta}\right)$ are those contained in equation (10). Other than this the relative sizes of ϵ , μ^2 and δ are arbitrary.

Another point of interest is that the analysis presented here does not preclude the possibility of reflected waves in η_1 , η_2 , and η_3 . A careful analysis of these terms shows that both right and left moving waves are possible in these solutions although the analysis presented here is only expected to give us the behavior of η_o and not that of the higher order terms.

In conclusion, the analysis presented here is somewhat more general than that of Johnson (1973). The derivation presented here is in terms of a standard technique of singular perturbation theory, namely multitime. The variable coefficient KdV equation (10) gives an accurate description of the flow for distances of order of the largest of ϵ^{-1} , μ^{-2} and δ^{-1} . Except for the difficulties mentioned earlier in this section there are no other restrictions of the size of ϵ , μ^2 and δ relative to each other.

This work was performed at the Naval Research Laboratory while the author participated in the ASEE Summer Faculty Research Program. Many useful discussions with James Witting formed part of the basis of this work; his support is gratefully acknowledged.

REFERENCES

- Cole, J. D., 1968. *Perturbation Methods in Applied Mathematics*, Blaisdell.
- Grimshaw, R., 1970. *J. Fl. Mech.* v. 42, pp. 401-409.
- Johnson, R.S., 1973. *Proc. Camb. Phil. Soc.* v. 73, pp. 183-203.
- Korteweg, D. J. and de Vries, G., 1895. *Philos. Mag.* v.39, pp. 422-443.
- Lamb, H. 1932. *Hydrodynamics*, Dover.
- Leibovich, S. and Seebass, A. R., 1974. *Nonlinear Waves*, Cornell University Press.
- Madsen, O. S. and Mei, C. C., 1969. *J. Fl. Mech.* v. 39, pp. 781-791.
- Nayfeh, A. H. 1974, *Perturbation Methods*, Wiley-Interscience.
- Ostrovskiy, L. A. and Pelinovskiy, E. N., 1970 *Atmospheric and Ocean Physics.* v. 6, pp. 934-939.
- Peregrine, D. H., 1967 *J. Fl. Mech.*, v. 27, pp. 815-827.
- Svendsen, I. A. and Hansen, J. B., 1978 *J. Fl. Mech.* v. 87, pp. 433-448.
- Van Dyke, M., 1975 *Perturbation Methods in Fluid Mechanics*, Parabolic Press.

Appendix A

Equations Resulting from Perturbation Expansions

The perturbation procedure outlined in Section 3 leads to the following system of equations:

$$\begin{aligned} \phi_{0yy} &= 0 \\ \phi_{0y} &= 0 & \text{on } y = -h & \quad (A1) \\ \left. \begin{aligned} \phi_{0y} &= 0 \\ \eta_0 &= \phi_{0\tau} \end{aligned} \right\} & \text{on } y = 0 \end{aligned}$$

$$\begin{aligned} \phi_{1yy} &= -[\dot{g}^2 \phi_{0\tau\tau} + 2\dot{g}\phi_{0x_0\tau} + \phi_{0x_0x_0}] \\ \phi_{1y} &= 0 & \text{on } y = -h & \quad (A2) \\ \left. \begin{aligned} \phi_{1y} &= -\eta_{0\tau} \\ \eta_1 &= \phi_{1\tau} \end{aligned} \right\} & \text{on } y = 0 \end{aligned}$$

$$\begin{aligned} \phi_{2yy} &= 0 \\ \phi_{2y} &= 0 & \text{on } y = -h & \quad (A3) \\ \left. \begin{aligned} \phi_{2y} &= 0 \\ \eta_2 &= \phi_{2\tau} \end{aligned} \right\} & \text{on } y = 0 \end{aligned}$$

$$\begin{aligned} \phi_{3yy} &= 0 \\ \phi_{3y} &= 0 & \text{on } y = -h & \quad (A4) \\ \left. \begin{aligned} \phi_{3y} &= 0 \\ \eta_3 &= \phi_{3\tau} - \frac{1}{2} (\phi_{0\tau}\dot{g} + \phi_{0x_0})^2 \end{aligned} \right\} & \text{on } y = 0 \end{aligned}$$

$$\begin{aligned} \phi_{4yy} &= -[\dot{g}^2 \phi_{1\tau\tau} + 2\dot{g}\phi_{1x_0\tau} + \phi_{1x_0x_0} + 2\dot{g}\phi_{0x_1\tau} + 2\phi_{0x_0x_1}] \\ \phi_{4y} &= 0 & \text{on } y = -h & \quad (A5) \\ \left. \begin{aligned} \phi_{4y} &= -\eta_{1\tau} \\ \eta_4 &= \phi_{4\tau} \end{aligned} \right\} & \text{on } y = 0 \end{aligned}$$

$$\begin{aligned} \phi_{5yy} &= -[\dot{g}^2 \phi_{2\tau\tau} + 2\dot{g}\phi_{2x_0\tau} + \phi_{2x_0x_0} + \ddot{g}\phi_{0\tau} + 2\dot{g}\phi_{0x_1\tau} + 2\phi_{0x_0x_1}] \\ \phi_{5y} &= -[\dot{g}\phi_{0\tau} + \phi_{0x_0}] h'(x_2) & \text{on } y = -h & \quad (A6) \end{aligned}$$

$$\left. \begin{aligned} \phi_{5y} &= -\eta_{2\tau} \\ \eta_5 &= \phi_{5\tau} \end{aligned} \right\} \quad \text{on } y = 0$$

$$\begin{aligned} \phi_{6yy} &= 0 \\ \phi_{6y} &= 0 \\ \phi_{6y} &= 0 \\ \eta_6 &= \phi_{6\tau} \end{aligned} \quad \left. \begin{aligned} &\text{on } y = -h \\ &\text{on } y = 0 \end{aligned} \right\} \quad (\text{A7})$$

$$\begin{aligned} \phi_{7yy} &= -[\dot{g}^2 \phi_{3\tau\tau} + 2\dot{g} \phi_{3x_u\tau} + \phi_{3x_u x_u} + 2\dot{g} \phi_{ax_3\tau} + 2\phi_{ax_3 x_u}] \\ \phi_{7y} &= 0 \\ \phi_{7y} &= -\eta_a \phi_{1yy} - \eta_{3\tau} + (\dot{g} \eta_{a\tau} + \eta_{ax_u}) (\dot{g} \phi_{a\tau} + \phi_{ax_u}) \\ -\eta_7 &= -\phi_{7\tau} - \eta_a \phi_{1y\tau} + \phi_{a\tau} \phi_{1\tau} + \phi_{ax_u} \phi_{1x_u} + \frac{1}{2} \phi_{1x_u} \phi_{a\tau} + \frac{1}{2} \phi_{1\tau} \phi_{ax_u} \\ &\quad + \phi_{ax_1} \dot{g} \phi_{a\tau} + \phi_{ax_1} \phi_{ax_u} + \frac{1}{2} \phi_{1y}^2 \end{aligned} \quad \left. \begin{aligned} &\text{on } y = -h \\ &\text{on } y = 0 \end{aligned} \right\} \quad (\text{A8})$$

$$\begin{aligned} \phi_{8yy} &= 0 \\ \phi_{8y} &= 0 \\ \phi_{8y} &= 0 \\ -\eta_8 &= -\phi_{8\tau} - \eta_a \phi_{2y\tau} + \phi_{a\tau} \phi_{2\tau} + \phi_{2x_u} \phi_{ax_u} + \frac{1}{2} \phi_{2\tau} \phi_{ax_u} \\ &\quad + \frac{1}{2} \phi_{2x_u} \phi_{a\tau} + \phi_{ax_2} \phi_{a\tau} \dot{g} + \phi_{ax_2} \phi_{ax_u} \end{aligned} \quad \left. \begin{aligned} &\text{on } y = -h \\ &\text{on } y = 0 \end{aligned} \right\} \quad (\text{A9})$$

$$\begin{aligned} \phi_{9yy} &= 0 \\ \phi_{9y} &= 0 \\ \phi_{9y} &= 0 \\ -\eta_9 &= -\phi_{9\tau} + \phi_{3\tau} \phi_{a\tau} + \phi_{3x_u} \phi_{ax_u} + \frac{1}{2} \phi_{3\tau} \phi_{ax_u} \\ &\quad + \frac{1}{2} \phi_{a\tau} \phi_{3x_u} + \phi_{ax_1} \dot{g} \phi_{a\tau} + \phi_{ax_1} \phi_{ax_u} \end{aligned} \quad \left. \begin{aligned} &\text{on } y = -h \\ &\text{on } y = 0. \end{aligned} \right\} \quad (\text{A10})$$

In these equations we have anticipated and used the result that $\phi_a \neq \phi_a(y)$ and in equations (A9) and (A10) we have used the result that $\phi_{2y} = \phi_{3y} \equiv 0$.

MARK S. CRAMER

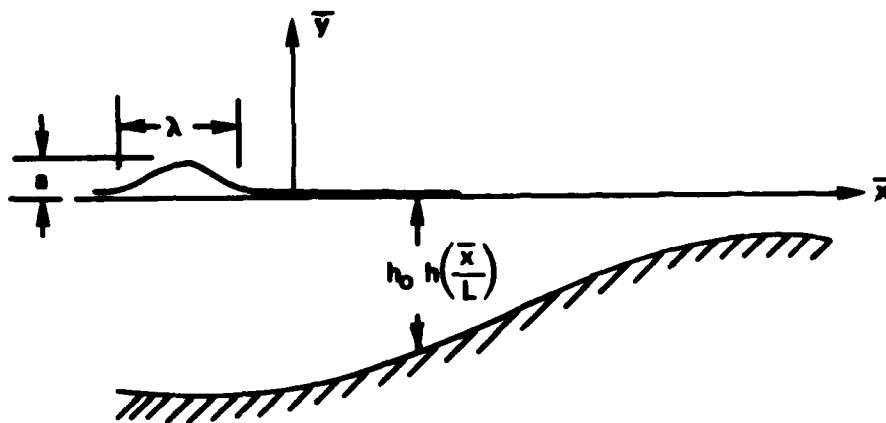


Fig. 1 — Notation and Coordinate System. The undisturbed depth is $\bar{y} = -h_0 h\left(\frac{\bar{x}}{L}\right)$ and a typical disturbance has wavelength λ and amplitude a ; $a \ll h_0 \ll \lambda \ll L$.

